Additional Exercises
CS 2210 Logic for Computer Scientists - Fall 2016
Try working on this as exam preparation. Feel free to discuss with Aaron during help-desk hours

Exercise 36 Give a structure for the formula

$$\forall x \forall y (Q(x, y) \rightarrow Q(y, x)).$$

Exercise 37

$$\begin{aligned} &\forall x(\text{human}(x) \rightarrow \text{parentOf}(\text{fatherOf}(x), x)) \\ &\wedge \forall x \forall y(\text{orphan}(x) \wedge \text{parentOf}(y, x) \rightarrow \text{dead}(y)) \end{aligned}$$

Give two structures for the above formula, one of which is a model for the formula, and one of which is not a model for the formula.

Exercise 38 Consider a structure $(U_{\mathcal{B}}, I_{\mathcal{B}})$:

$$\begin{aligned} U_{\mathcal{B}} &= \{\text{☺}, \text{☹}\} \\ a^{\mathcal{B}} &= \text{☺} \\ s^{\mathcal{B}} &: \text{☺} \mapsto \text{☹}; \text{☹} \mapsto \text{☹} \\ P^{\mathcal{B}} &= U_{\mathcal{B}} \\ Q^{\mathcal{B}} &= \{(\text{☹}, \text{☹})\} \end{aligned}$$

Show that $(U_{\mathcal{B}}, I_{\mathcal{B}})$ is a model for the formula:

$$\forall x \exists y (P(x) \wedge Q(s(x), y)).$$

Exercise 39 Consider the following sentences:

1. Healthy beings are not dead.
2. Every cat is alive or dead.
3. If somebody owns something, (s)he cares for it.
4. A happy cat owner owns a cat and all beings he cares for are healthy.
5. Schroedinger is a happy cat owner.

Sentence 1 above can be written as.

$$\forall x(\text{Healthy}(x) \rightarrow \neg\text{Dead}(x)).$$

Translate all other sentences above into first-order predicate logic formula. Use schroedinger as a constant symbol and use only the following predicate symbols:
unary: Healthy, Dead, Cat, Alive, HappyCatOwner
binary: owns, cares

Exercise 40 Sketch how you could formally prove, using the formulas from Exercise 59, that Schroedinger's cat is alive.

Exercise 41 Show, that $\forall x \exists y P(x, y) \not\equiv \exists u \forall v P(v, u)$.

Exercise 42 Show, using the statements from Theorem 3.4.1, that $\forall x \exists y (P(x) \wedge Q(y)) \equiv \exists y \forall x (P(x) \wedge Q(y))$.

Exercise 43 Show by using the statements from of Theorem 3.4.1, that

$$\forall x (P(x) \rightarrow (\exists y (O(x,y) \wedge C(y)) \wedge (\forall z (R(x,z) \rightarrow H(z)))))$$

and

$$\forall z \forall x \exists y ((P(x) \rightarrow (O(x,y) \wedge C(y))) \wedge ((P(x) \wedge R(x,z)) \rightarrow H(z)))$$

are equivalent.

Exercise 44 What is $(\forall x (Q(x,y,z)[y/a])[x/b] \wedge \forall x (P(x,y)[y/x][x/a]))[z/x]$?

Exercise 45 Show, that, for any formula F in which y does not occur as free variable, $\forall x F \equiv \forall y F[x/y]$.

Exercise 46 Consider the following formulas:

1. $\forall x (H(x) \rightarrow \neg D(x))$
2. $\forall x (C(x) \rightarrow (A(x) \vee D(x)))$
3. $\forall x \forall y (O(x,y) \rightarrow R(x,y))$
4. $\forall x (P(x) \rightarrow (\exists y (O(x,y) \wedge C(y)) \wedge (\forall y (R(x,y) \rightarrow H(y)))))$
5. $P(s)$

The symbol s in formula 5 above is a constant. Transform all formulas above into NNF.

Exercise 47 Show, using a tableau, that $\exists x (P(x) \wedge Q(x)) \models \exists x P(x) \wedge \exists y Q(y)$.

Exercise 48 Show, using a tableau, that $\exists x (O(s,x) \wedge A(x))$ is a logical consequence of the formulas in Exercise 46.

Exercise 49 Show, using a tableau, that $Q(a) \wedge Q(b) \wedge \forall x (P(x) \wedge (Q(x) \rightarrow \neg P(x)))$ is unsatisfiable.

Exercise 50 Identify all predicate symbols and all terms in the formula 4 of Exercise 46.

Exercise 51 Determine all predicate symbols and all function symbols, with arities, of the following formulas:

$$\forall x \forall y (x + y = y + x) \tag{1}$$

$$\forall x (\text{prime}(x) \leftrightarrow \neg \exists n (n \in \mathbb{N} \wedge n > 1 \wedge \frac{x}{n} \in \mathbb{N})) \tag{2}$$

$$\exists \delta \exists x ((2^4 \neq \delta \rightarrow <(|38|, 75, 3)) \rightarrow \forall \phi (\phi = a \vee >(\phi))) \tag{3}$$

$$\forall x \forall z \exists y (R(\varepsilon(x), g(y,z)) \rightarrow A \vee (B(f(g(x,y))) \wedge R(\delta(\sigma(a, bc))))) \tag{4}$$

Exercise 52 Give all subformulas of formula 1, 2, and 3 from Exercise 46. Which of them are closed? Which of them are open?

Exercise 53 Give all subformulas of the formulas in Exercise 51. Which of them are closed? Which of them are open?

Exercise 54 Formula (1) in Exercise 51 expresses the commutative law of addition. Similar to that formula, how would you write the following as formulas?

(a) The commutative law of multiplication.

(b) The associative law of addition.

(c) The distributive law of addition and multiplication.

(d) Convergence of a sequence of real numbers to a limit.

Exercise 55 Give a structure for each one of the following formulas:

$$\forall w \forall v \exists z (R(w, v, w) \rightarrow B(z, z)) \tag{5}$$

$$\forall x \forall y \exists z (A(x, y) \vee B(z, z) \wedge C(x, y, x)) \tag{6}$$

Exercise 56 For formulas (1) and (2) of Exercise 51 as well as all formulas you obtained as answer to Exercise 54 (a), (b), and (c), give two structures, one of which is a model and the other is not a model.

Exercise 57 Give two structures for each fo the formulas in Exercise 55. For every formula, exactly one of the structures should be a model.

Exercise 58 Determine whether $(U_{\mathcal{B}}, I_{\mathcal{B}})$ as defined in Exercise 38 is a model for the following formulas:

$$P(a) \wedge Q(s(s(a)), s(a)) \tag{7}$$

$$\forall x \exists y (Q(y, s(x)) \wedge P(y) \wedge P(s(x))) \tag{8}$$

Justify your answers!

Exercise 59 Translate the following sentences into first-order logic:

1. David is friends with Alice.
2. Everyone who studies logics is cool.
3. Everyone with cool friends is also cool.
4. Everybody who is friends with a bunny is nuts.
5. Alice is friends with some bunny.
6. Friendship is a transitive and symmetric relationship!
7. Everybody who eats ramen is either homeless or a grad student.
8. David eats ramen, is not homeless and studies logics.

Use the following symbols:

- Unary predicates: Student, Nuts, GradStudent, Ramen, Bunny, Homeless, Logics, Cool

- Binary predicates: Studies, IsFriendsWith, Eats,
- Constants: david, alice

Exercise 60 Sketch how you could formally prove, using the formulas from Exercise 59, that David is both nuts and a grad student; and that somewhere, there must be some cool bunny!

Exercise 61 Show the following without using any of the statements in Theorem 3.4.1:

$$F \lor Q \equiv \neg(\neg Q \land \neg F)$$
$$(Q \land F) \lor G \equiv (G \lor Q) \land (F \lor Q)$$

Exercise 62 Show the following:

$$\forall x \exists y (P(x) \land Q(y)) \not\equiv \exists u \exists v (P(v) \land Q(u))$$
$$\forall x P(x) \land \forall x Q(x) \not\equiv \forall x (P(x) \land Q(x))$$

Exercise 63 Show the following equivalences using the statements from Theorem 3.4.1:

$$\neg \forall x (P(x) \lor Q(x)) \equiv \exists x \neg Q(x) \land \exists x \neg P(x)$$
$$\forall x (P(x) \land \neg \exists y \forall z Q(y, z)) \equiv \forall y \exists z (\neg Q(y, z) \land \forall x P(x))$$

Exercise 64 Complete the following:

$$(\forall x P(x, x) \land \exists y \forall z R(y, y, z))[x/a][a/c][x/d] = \dots$$
$$((\exists y \forall z \forall x Q(y, z))[z/a][a/b] \land \forall z R(z, z, x))[y/x][x, b] = \dots$$

Exercise 65 Show the following:

(a) For any formula F without occurrences of the variable x, $\exists z F \equiv \exists x F[z/x]$.

(b) $\exists y F$ and $F[y/a]$, where a is some constant, are equisatisfiable.

Exercise 66 Transform all formulas from Exercise 54 into NNF.

Exercise 67 Show the following using a tableau:

$$\exists y (P(y) \lor Q(y)) \land \forall x R(x) \land \forall x (R(x) \to \neg Q(x)) \models \exists x P(x)$$
$$\exists y (P(y) \land \neg S(y)) \land \forall y (\neg P(y) \lor Q(y)) \land \forall y (\neg Q(y) \lor (R(y) \land S(y))) \models \exists x (P(x) \land R(x))$$

Exercise 68 Show the following using a tableau:

(a) $\exists y \text{Bunny}(y) \land \text{Cool}(y)$ is a consequence of formulas 1, 2, 3, 5, 6 and 8 from Exercise 54.

(b) $\text{Nuts(david)} \land \text{GradStudent(david)}$ is a consequence of formulas 1, 4, 5, 6, 7 and 8 from Exercise 54.

Exercise 69 Show, using a tableau, whether or not the following formulas are satisfiable:

$$Q(a) \land R(a) \land \forall x (R(x) \to (P(x, x) \lor \neg Q(x))) \land \forall y (P(y, y) \land \neg Q(y))$$
$$R(a, b) \land \forall x (R(x, b) \to (\neg R(x, x) \land B(x))) \land \forall x (R(x, x) \lor \neg B(x))$$