Solutions for Exercise Sheet 6
CS 2210 Logic for Computer Scientists - Spring 2016

Exercise 37 Identify all predicate symbols (with their arity) and all function symbols (with their arity) in all of the following formulas where s is a constant:

$$\forall x(P(x) \rightarrow \exists y(Q(f(x), s) \vee \neg Q(y, y)))$$
$$\exists x A(x, s) \wedge \neg\exists y(Q(y, x) \vee \neg P(y, s))$$
$$\forall y(\exists x R(x) \rightarrow \exists x(\neg Q(y, p(x, z), s)))$$
$$\exists x \neg R(x) \wedge \forall x Q(x, g(g(x)))$$

Solution:
We start with the first formula $\forall x(P(x) \rightarrow \exists y(Q(f(x), s) \vee \neg Q(y, y)))$.
The predicate symbols are P with arity 1 and Q with arity 2. The function symbols are f with arity 1 and s with arity 0. Note that the problem statement stated that s is a constant, therefore it is categorized as a function symbol (with arity 0). Note that all function symbols occur "inside" some predicate symbol.

Next the second formula: $\exists x A(x, s) \wedge \neg\exists y(Q(y, x) \vee \neg P(y, s))$.
The predicate symbols are A with arity 2, Q with arity 2, and P with arity 2. The function symbol is s with arity 0.

The third formula: $\forall y(\exists x R(x) \rightarrow \exists x(\neg Q(y, p(x, z), s)))$.
The predicate symbols are R with arity 1 and Q with arity 3. The function symbols are p with arity 2 and s with arity 0.

The fourth formula: $\exists x \neg R(x) \wedge \forall x Q(x, g(g(x)))$.
The predicate symbols are R with arity 1 and Q with arity 2. The funcrion symbol is g with arity 1. Note that here a function symbol may occur "inside" another (possibly the same) function symbol as long as according to its arity. In contrast, a predicate symbol must not occur "inside" another predicate symbol or function symbol.

Exercise 38 Give all terms in all of the formulas in Exercise 37.

Solution:
Recall that t is a term if t is a variable or t is of the form $f(t_1, \ldots, t_k)$ where f is a function symbol of arity k and $t_1, \ldots, t_k$ are also terms. Notably, a constant, which is a function symbol of arity 0, is also a term.

The first formula: $\forall x(P(x) \rightarrow \exists y(Q(f(x), s) \vee \neg Q(y, y)))$.
The terms include x, y, s, and $f(x)$.

The second formula: $\exists x A(x, s) \wedge \neg\exists y(Q(y, x) \vee \neg P(y, s))$.
The terms include x, y, and s.

The third formula: $\forall y(\exists x R(x) \rightarrow \exists x(\neg Q(y, p(x, z), s)))$.
The terms are x, y, z, s, and $p(x, z)$. Note that in the problem statement, only s is a constant, so z is a variable.

The fourth formula: $\exists x \neg R(x) \wedge \forall x Q(x, g(g(x)))$.
The terms are x, $g(x)$, and $g(g(x))$.

Exercise 39 Give all subformulas of each of the formulas in Exercise 37.

Solution:
The first formula: $\forall x(P(x) \to \exists y(Q(f(x), s) \vee \neg Q(y, y)))$.
Note that the above formulas is actually $\forall x(\neg P(x) \vee \exists y(Q(f(x), s) \vee \neg Q(y, y)))$
The subformulas of the above formula are $P(x)$, $\neg P(x)$, $Q(f(x), s)$, $Q(y, y)$, $\neg Q(y, y)$, $Q(f(x), s) \vee \neg Q(y, y)$, $\exists y(Q(f(x), s) \vee \neg Q(y, y))$, $\neg P(x) \vee \exists y(Q(f(x), s) \vee \neg Q(y, y))$, and $\forall x(\neg P(x) \vee \exists y(Q(f(x), s) \vee \neg Q(y, y)))$.

The second formula: $\exists x A(x, s) \wedge \neg \exists y(Q(y, x) \vee \neg P(y, s))$.
The subformulas are $A(x, s)$, $\exists x A(x, s)$, $Q(y, x)$, $P(y, s)$, $\neg P(y, s)$, $Q(y, x) \vee \neg P(y, s)$, $\exists y(Q(y, x) \vee \neg P(y, s))$, $\neg \exists y(Q(y, x) \vee \neg P(y, s))$, and $\exists x A(x, s) \wedge \neg \exists y(Q(y, x) \vee \neg P(y, s))$.

The third formula: $\forall y(\exists x R(x) \to \exists x(\neg Q(y, p(x, z), s)))$.
Rewriting the implication, the above formula is $\forall y(\neg \exists x R(x) \vee \exists x(\neg Q(y, p(x, z), s)))$.
The subformulas are $R(x)$, $\exists x R(x)$, $\neg \exists x R(x)$, $Q(y, p(x, z), s)$, $\neg Q(y, p(x, z), s)$, $\exists x(\neg Q(y, p(x, z), s))$, $\neg \exists x R(x) \vee \exists x(\neg Q(y, p(x, z), s))$, and $\forall y(\neg \exists x R(x) \vee \exists x(\neg Q(y, p(x, z), s)))$.

The fourth formula: $\exists x \neg R(x) \wedge \forall x Q(x, g(g(x)))$.
The subformulas are $R(x)$, $\neg R(x)$, $\exists x \neg R(x)$, $Q(x, g(g(x)))$, $\forall x Q(x, g(g(x)))$, and $\exists x \neg R(x) \wedge \forall x Q(x, g(g(x)))$.

Exercise 40 For each of the formulas in Exercise 37, determine which variables are bound and which are free. Based on your answer, also determine if the formula is closed or open.
Solution:
The first formula: $\forall x(P(x) \to \exists y(Q(f(x), s) \vee \neg Q(y, y)))$.
In the above formula, both variables x and y are bound. So, the formulas is closed.

The second formula: $\exists x A(x, s) \wedge \neg \exists y(Q(y, x) \vee \neg P(y, s))$.
In the above formulas, variable y is bound while the variable x is both both bound and free. It is bound when it occurs in $\exists x A(x, s)$, and it is free when it occurs in $\exists y(Q(y, x) \vee \neg P(y, s))$. Since there is at least one free occurrence of variables, the formula is open.

The third formula: $\forall y(\exists x R(x) \to \exists x(\neg Q(y, p(x, z), s)))$.
In the above formula, the variables y and x are bound. So the formula is closed.

The fourth formula: $\exists x \neg R(x) \wedge \forall x Q(x, g(g(x)))$.
Above, the variable x is bound, hence the formula is closed. Note that the first occurrence of x is bound by an existential quantifier, while the second and third occurrences are bound a universal quantifier.

Exercise 41 Give a structure for the formula

$$\forall x \forall y (Q(x, y) \to Q(y, x)).$$

Determine whether the structure you gave is a model of the formula.

Solution:
The formula above only has one predicate symbol, namely Q with arity 2, and no function symbol. An example of a structure of the formula is $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ where the universe $U_{\mathcal{A}}$ and the mapping $I_{\mathcal{A}}$ are defined below.

$$U_{\mathcal{A}} = \{1, 2, 3, \dots\}$$
$$Q^{\mathcal{A}} = \{(1, 2), (2, 1)\}$$

We determine whether $\mathcal{A}$ is a model of the formula, i.e., whether $\mathcal{A}(\forall x \forall y (Q(x, y) \to Q(y, x))) = 1$.

- Now, $\mathcal{A}(\forall x \forall y (Q(x, y) \to Q(y, x))) = 1$ if for every $x \in \{1, 2, 3, \dots\}$, it holds that $\mathcal{A}(\forall y (Q(x, y) \to Q(y, x))) = 1$. Otherwise, $\mathcal{A}(\forall x \forall y (Q(x, y) \to Q(y, x))) = 0$.

- We proceed with checking whether for every $x \in \{1, 2, 3, \dots\}$, $\mathcal{A}(\forall y (Q(x, y) \to Q(y, x))) = 1$. This is equivalent to having $\mathcal{A}(\forall y (Q(1, y) \to Q(y, 1))) = 1$ and $\mathcal{A}(\forall y (Q(2, y) \to Q(y, 2))) = 1$ and $\mathcal{A}(\forall y (Q(3, y) \to Q(y, 3))) = 1$ and so forth (i.e., substitute x with every element of the set $\{1, 2, 3, \dots\}$.

- Thus, we first check for the case of $x = 1$, i.e., if $\mathcal{A}(\forall y (Q(1, y) \to Q(y, 1))) = 1$. Here, $\mathcal{A}(\forall y (Q(1, y) \to Q(y, 1))) = 1$ if for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(Q(1, y) \to Q(y, 1)) = 1$. That is, if for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(\neg Q(1, y) \lor Q(y, 1)) = 1$.
  This is equivalent to having that for all $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(\neg Q(1, y)) = 1$ or $\mathcal{A}(Q(y, 1)) = 1$. In other words, we check if for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(Q(1, y)) = 0$ or $\mathcal{A}(Q(y, 1)) = 1$. We proceed by checking the last statement on every y.
  For $y = 1$, we have that $\mathcal{A}(Q(1, y)) = 0$ because $(1, 1) \notin Q^{\mathcal{A}}$. So, the statement holds.
  For $y = 2$, we have that $\mathcal{A}(Q(1, y)) = 1$ and $\mathcal{A}(Q(y, 1)) = 1$ because $(1, 2), (2, 1) \in Q^{\mathcal{A}}$. So, the statement again holds.
  For $y = 3$, we have that $\mathcal{A}(Q(1, y)) = 0$, hence the statement holds.
  We continue this for every y, and we shall always find that the statement holds. Therefore, we can infer that for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(Q(1, y) \to Q(y, 1)) = 1$.

- The same thing as above is done for the case of $x = 2$, and we shall find that for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(Q(2, y) \to Q(y, 2)) = 1$ because $\mathcal{A}(Q(2, y)) = 0$ for all $y \neq 1$ and $\mathcal{A}(Q(y, 2)) = 1$ when $y = 1$.

- For the case of $x = 3$, we can infer that for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(Q(3, y) \to Q(y, 3)) = 1$ because $\mathcal{A}(Q(3, y)) = 0$ for every $y \in \{1, 2, 3, \dots\}$. The cases for $x = 4, 5, \dots$ are the same.

- Thus, for every $x \in \{1, 2, 3, \dots\}$ it holds that for every $y \in \{1, 2, 3, \dots\}$, $\mathcal{A}(Q(x, y) \to Q(y, x)) = 1$. That is, for every $x \in \{1, 2, 3, \dots\}$, $\mathcal{A}(\forall y (Q(x, y) \to Q(y, x))) = 1$.

- So, we conclude that $\mathcal{A}(\forall x \forall y (Q(x, y) \to Q(y, x))) = 1$.

The whole narration above can be more concisely written as follows:

$$
\begin{aligned}
&\mathcal{A}(\forall x \forall y(Q(x,y) \to Q(y,x)))\\
&= \begin{cases} 1, & \text{if for every } x \in \{1,2,3,\dots\}, \mathcal{A}(\forall y(Q(x,y) \to Q(y,x))) = 1\\ 0, & \text{otherwise} \end{cases}\\
&= \begin{cases} 1, & \text{if for every } x \in \{1,2,3,\dots\} \text{ it holds that for every } y \in \{1,2,3,\dots\}, \mathcal{A}(Q(x,y) \to Q(y,x)) = 1\\ 0, & \text{otherwise} \end{cases}\\
&= \begin{cases} 1, & \text{if for every } x \in \{1,2,3,\dots\} \text{ and for every } y \in \{1,2,3,\dots\}, \mathcal{A}(\neg Q(x,y) \vee Q(y,x)) = 1\\ 0, & \text{otherwise} \end{cases}\\
&= \begin{cases} 1, & \text{if for every } x \in \{1,2,3,\dots\} \text{ and for every } y \in \{1,2,3,\dots\}, \mathcal{A}(\neg Q(x,y)) = 1 \text{ or } \mathcal{A}(Q(y,x)) = 1\\ 0, & \text{otherwise} \end{cases}\\
&= \begin{cases} 1, & \text{if for every } x \in \{1,2,3,\dots\} \text{ and for every } y \in \{1,2,3,\dots\}, \mathcal{A}(Q(x,y)) = 0 \text{ or } \mathcal{A}(Q(y,x)) = 1\\ 0, & \text{otherwise} \end{cases}\\
&= 1,\ \text{because (i) when } x = 0 \text{ and } x > 2,\ \text{we have for every } y \text{ that } (x,y) \notin Q^{\mathcal{A}},\ \text{i.e., } \mathcal{A}(Q(x,y)) = 0;\\
&\qquad \text{(ii) when } x = 1 \text{ and } y \neq 2,\ \text{we have } (x,y) \notin Q^{\mathcal{A}},\ \text{i.e., } \mathcal{A}(Q(x,y)) = 0;\\
&\qquad \text{(iii) when } x = 1 \text{ and } y = 2,\ \text{we have } (y,x) \in Q^{\mathcal{A}},\ \text{i.e., } \mathcal{A}(Q(y,x)) = \mathcal{A}(Q(2,1)) = 1;\\
&\qquad \text{(iv) when } x = 2 \text{ and } y \neq 1 \text{ we have } (x,y) \notin Q^{\mathcal{A}},\ \text{i.e., } \mathcal{A}(Q(x,y)) = 0;\\
&\qquad \text{(v) when } x = 2 \text{ and } y = 1 \text{ we have } (y,x) \in Q^{\mathcal{A}},\ \text{i.e., } \mathcal{A}(Q(y,x)) = \mathcal{A}(Q(1,2)) = 1.
\end{aligned}
$$

So, the structure is a model of the given formula.

Exercise 42 Give two structures for the following formula, one of which is a model for the formula, and the other is not a model for the formula.

$$\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x)) \land \forall x \forall y(\text{orphan}(x) \land \text{parentOf}(y, x) \to \text{dead}(y))$$

Solution:
Let F be the above formula. The formula has the following predicates: human with arity 1, parentOf with arity 2, orphan with arity 1, and dead with arity 1; and the following function symbols: fatherOf with arity 1.
The first structure is $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ where

$$\begin{aligned}
U_\mathcal{A} &= \{a\} \\
\text{fatherOf}^\mathcal{A} &: a \mapsto a, \quad \text{note that fatherOf is a unary function} \\
\text{human}^\mathcal{A} &= \{a\} \\
\text{parentOf}^\mathcal{A} &= \{(a, a)\} \\
\text{orphan}^\mathcal{A} &= \{a\} \\
\text{dead}^\mathcal{A} &= \{a\}
\end{aligned}$$

We evaluate $\mathcal{A}(F)$ below.
$\mathcal{A}(F) = 1$ if and only if both $\mathcal{A}(\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x))) = 1$ and $\mathcal{A}(\forall x \forall y(\text{orphan}(x) \land \text{parentOf}(y, x) \to \text{dead}(y))) = 1$.

We first check if $\mathcal{A}(\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x))) = 1$.

$$\begin{aligned}
&\mathcal{A}(\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x))) \\
&= \begin{cases} 1, & \text{if for all } x \in \{a\}, \mathcal{A}(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{A}(\text{human}(a) \to \text{parentOf}(\text{fatherOf}(a), a)) = 1 \quad \text{note: } U_\mathcal{A} \text{ only has one element } a \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{A}(\text{human}(a) \to \text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{A}(\neg\text{human}(a) \lor \text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{A}(\neg\text{human}(a)) = 1 \text{ or } \mathcal{A}(\text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{A}(\text{human}(a)) = 0 \text{ or } \mathcal{A}(\text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= 1, \qquad \text{because } (a, a) \in \text{parentOf}^\mathcal{A} \text{ making } \mathcal{A}(\text{parentOf}(a, a)) = 1
\end{aligned}$$

Next, we check if $\mathcal{A}(\forall x \forall y(\text{orphan}(x) \wedge \text{parentOf}(y, x) \rightarrow \text{dead}(y))) = 1$.

$$
\begin{aligned}
&\mathcal{A}(\forall x \forall y(\text{orphan}(x) \wedge \text{parentOf}(y, x) \rightarrow \text{dead}(y))) \\
&= \begin{cases} 1, & \text{if for all } x \in \{a\} \text{ and for all } y \in \{a\}, \mathcal{A}(\text{orphan}(x) \wedge \text{parentOf}(y, x) \rightarrow \text{dead}(y)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{A}(\text{orphan}(a) \wedge \text{parentOf}(a, a) \rightarrow \text{dead}(a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= 1, \qquad \text{because } \mathcal{A}(\text{orphan}(a)) = 1 \text{ due to the fact that } a \in \text{orphan}^{\mathcal{A}}, \\
&\qquad\quad \mathcal{A}(\text{parentOf}(a, a)) = 1 \text{ due to the fact that } (a, a) \in \text{parentOf}^{\mathcal{A}}, \textit{and} \\
&\qquad\quad \mathcal{A}(\text{dead}(a)) = 1 \text{ due to the fact that } a \in \text{dead}^{\mathcal{A}}.
\end{aligned}
$$

Since both $\mathcal{A}(\forall x(\text{human}(x) \rightarrow \text{parentOf}(\text{fatherOf}(x), x))) = 1$ and $\mathcal{A}(\forall x \forall y(\text{orphan}(x) \wedge \text{parentOf}(y, x) \rightarrow \text{dead}(y))) = 1$, we conclude that $\mathcal{A}(F) = 1$, i.e., $\mathcal{A}$ is a model for F.

Now for the second structure, we have $\mathcal{B} = (U_{\mathcal{B}}, I_{\mathcal{B}})$ where

$$
\begin{aligned}
U_{\mathcal{B}} &= \{a\} \\
\text{fatherOf}^{\mathcal{B}} &: a \mapsto a \\
\text{human}^{\mathcal{B}} &= \{a\} \\
\text{parentOf}^{\mathcal{B}} &= \emptyset \\
\text{orphan}^{\mathcal{B}} &= \{a\} \\
\text{dead}^{\mathcal{B}} &= \{a\}
\end{aligned}
$$

We shall see that $\mathcal{B}$ is not model for F, i.e., $\mathcal{B}(F) = 0$. To show this, we need to show that $\mathcal{B}(\forall x(\text{human}(x) \rightarrow \text{parentOf}(\text{fatherOf}(x), x))) = 0$ or $\mathcal{B}(\forall x \forall y(\text{orphan}(x) \wedge \text{parentOf}(y, x) \rightarrow \text{dead}(y))) = 0$.

We show that $\mathcal{B}(\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x))) = 0$.

$$
\begin{aligned}
&\mathcal{B}(\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x))) \\
&= \begin{cases} 1, & \text{if for all } x \in \{a\}, \mathcal{B}(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{B}(\text{human}(a) \to \text{parentOf}(\text{fatherOf}(a), a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{B}(\text{human}(a) \to \text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{B}(\neg\text{human}(a) \vee \text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{B}(\neg\text{human}(a)) = 1 \text{ or } \mathcal{B}(\text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \mathcal{B}(\text{human}(a)) = 0 \text{ or } \mathcal{B}(\text{parentOf}(a, a)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= 0, \qquad \text{because } \mathcal{B}(\text{human}(a)) = 1 \text{ [as } a \in \text{human}^{\mathcal{B}}\text{] and } \mathcal{B}(\text{parentOf}(a, a)) = 0 \text{ [as } (a, a) \notin \text{parentOf}^{\mathcal{B}}\text{]}
\end{aligned}
$$

Notice that the above computation is very similar to the one we did for $\mathcal{A}$. The difference only lies on the last step.

Since $\mathcal{B}(\forall x(\text{human}(x) \to \text{parentOf}(\text{fatherOf}(x), x))) = 0$, we infer that $\mathcal{B}(F) = 0$, i.e., $\mathcal{B}$ is not a model for F.

Exercise 43 Give two structures for the following formula, one of which is a model for the formula and the other is not a model for the formula.

$$\forall x \exists y (P(x) \wedge Q(s(x), y)).$$

Solution:
Let the above formula be F. The formula has predicates P of arity 1 and Q of arity 2, and function symbol s of arity 1.

The first structure is $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ where

$$\begin{aligned} U_{\mathcal{A}} &= \mathbb{N} = \{0, 1, 2, \dots\} \\ s^{\mathcal{A}} &: n \mapsto n + 1 \\ P^{\mathcal{A}} &= \mathbb{N} = \{0, 1, 2, \dots\} \\ Q^{\mathcal{A}} &= \{(m, n) \mid m = n\} \end{aligned}$$

We shall show that $\mathcal{A}$ is a model for F.

$$\begin{aligned} &\mathcal{A}(\forall x \exists y (P(x) \wedge Q(s(x), y))) \\ &= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } \mathcal{A}(P(x) \wedge Q(s(x), y)) = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } \mathcal{A}(P(x)) = 1 \text{ and } \mathcal{A}(Q(s(x), y)) = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } \mathcal{A}(P(x)) = 1 \text{ and } \mathcal{A}(Q(x+1, y)) \\ & \text{[Note: } s^{\mathcal{A}}(x) = x + 1] \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } x \in \mathbb{N} \text{ and } x + 1 = y \\ & \text{[Note: } P^{\mathcal{A}} = \mathbb{N}, \text{ so } \mathcal{A}(P(x)) = 1 \text{ iff } x \in \mathbb{N}, \\ & \ \text{also } \mathcal{A}(Q(x+1, y)) = 1 \text{ iff } (x+1, y) \in Q^{\mathcal{A}} \text{ iff } x + 1 = y] \\ 0, & \text{otherwise} \end{cases} \\ &= 1, \quad \text{because for every } x \in \mathbb{N}, \text{ obviously } x \in \mathbb{N}, \\ &\qquad\quad \text{and for each such } x, \text{ there is clearly a } y \in \mathbb{N} \text{ with } y = x + 1 \end{aligned}$$

The second structure is $\mathcal{B} = (U_{\mathcal{B}}, I_{\mathcal{B}})$ where

$$\begin{aligned} U_{\mathcal{B}} &= \mathbb{N} = \{0, 1, 2, \dots\} \\ s^{\mathcal{B}} &: n \mapsto n \\ P^{\mathcal{B}} &= \mathbb{N} = \{0, 1, 2, \dots\} \\ Q^{\mathcal{B}} &= \{(m, n) \mid m > n\} \end{aligned}$$

We show that $\mathcal{B}$ cannot be a model for F.

$$
\begin{aligned}
&\mathcal{B}(\forall x \exists y (P(x) \land Q(s(x), y))) \\
&= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } \mathcal{B}(P(x) \land Q(s(x), y)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } \mathcal{B}(P(x)) = 1 \text{ and } \mathcal{B}(Q(s(x), y)) = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } \mathcal{B}(P(x)) = 1 \text{ and } \mathcal{B}(Q(x, y)) \\ & [\text{Note: } s^{\mathcal{B}}(x) = x] \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if for every } x \in \mathbb{N} \text{ there is } y \in \mathbb{N} \text{ with } x \in \mathbb{N} \text{ and } x > y \\ & [\text{Note: } P^{\mathcal{B}} = \mathbb{N}, \text{ so } \mathcal{B}(P(x)) = 1 \text{ iff } x \in \mathbb{N}, \\ & \ \text{also } \mathcal{B}(Q(x, y)) = 1 \text{ iff } (x, y) \in Q^{\mathcal{B}} \text{ iff } x > y] \\ 0, & \text{otherwise} \end{cases} \\
&= 0, \quad \text{because there is an } x \in \mathbb{N}, \text{ namely } x = 0, \text{ such that there is no } y \in \mathbb{N} \text{ with } x > y
\end{aligned}
$$

Exercise 44 Give two different models of the following formula where oh is a constant.

$$\forall x(\text{locatedIn}(\text{capitalOf}(x), x)) \land \text{locatedIn}(\text{capitalOf}(\text{oh}), \text{oh})$$

Solution:
Let the above formula be F. It has a predicate locatedIn of arity 2 and two function symbols: capitalOf of arity 1 and oh of arity 0.

A structure for F is $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ where

$$\begin{aligned} U_{\mathcal{A}} &= \{\text{ohio}, \text{columbus}\} \\ \text{oh}^{\mathcal{A}} &= \text{ohio} \\ \text{capitalOf}^{\mathcal{A}} &: \text{ohio} \mapsto \text{columbus}, \text{columbus} \mapsto \text{columbus} \\ \text{locatedIn}^{\mathcal{A}} &= \{(\text{columbus}, \text{ohio}), (\text{columbus}, \text{columbus})\} \end{aligned}$$

We verify that $\mathcal{A}$ is a model for F.

$$\mathcal{A}(\forall x(\text{locatedIn}(\text{capitalOf}(x), x)) \land \text{locatedIn}(\text{capitalOf}(\text{oh}), \text{oh}))$$

$$= \begin{cases} 1, & \text{if } \mathcal{A}(\forall x(\text{locatedIn}(\text{capitalOf}(x), x))) = 1 \text{ and } \mathcal{A}(\text{locatedIn}(\text{capitalOf}(\text{oh}), \text{oh})) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if for every } x \in \{\text{ohio}, \text{columbus}\}, \mathcal{A}(\text{locatedIn}(\text{capitalOf}(x), x)) = 1 \text{ and} \\ & \mathcal{A}(\text{locatedIn}(\text{capitalOf}(\text{ohio}), \text{ohio})) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if } \mathcal{A}(\text{locatedIn}(\text{capitalOf}(\text{ohio}), \text{ohio})) = 1 \text{ and } \mathcal{A}(\text{locatedIn}(\text{capitalOf}(\text{columbus}), \text{columbus})) = 1 \\ & \text{and } \mathcal{A}(\text{locatedIn}(\text{capitalOf}(\text{ohio}), \text{ohio})) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if } \mathcal{A}(\text{locatedIn}(\text{columbus}, \text{ohio})) = 1 \text{ and } \mathcal{A}(\text{locatedIn}(\text{columbus}, \text{columbus})) = 1 \\ & \text{and } \mathcal{A}(\text{locatedIn}(\text{columbus}, \text{ohio})) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= 1, \qquad \text{because } (\text{columbus}, \text{ohio}) \in \text{locatedIn}^{\mathcal{A}} \text{ and } (\text{columbus}, \text{columbus}) \in \text{locatedIn}^{\mathcal{A}}$$

For the second structure, we have $\mathcal{B} = (U_{\mathcal{B}}, I_{\mathcal{B}})$ where

$$\begin{aligned} U_{\mathcal{B}} &= \mathbb{N} = \{0, 1, 2, \dots\} \\ \text{oh}^{\mathcal{B}} &= 1 \\ \text{capitalOf}^{\mathcal{B}} &: n \mapsto 2n \\ \text{locatedIn}^{\mathcal{B}} &= \{(m, n) \mid m = 2n\} \end{aligned}$$

We verify that $\mathcal{B}$ is also a model for F.

$$\mathcal{B}(\forall x(\text{locatedIn}(\text{capitalOf}(x), x)) \wedge \text{locatedIn}(\text{capitalOf}(\text{oh}), \text{oh}))$$
$$= \begin{cases} 1, & \text{if } \mathcal{B}(\forall x(\text{locatedIn}(\text{capitalOf}(x), x))) = 1 \text{ and } \mathcal{B}(\text{locatedIn}(\text{capitalOf}(\text{oh}), \text{oh})) = 1 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & \text{if for every } x \in \mathbb{N},\ \mathcal{B}(\text{locatedIn}(\text{capitalOf}(x), x)) = 1,\ \text{and } \mathcal{B}(\text{locatedIn}(\text{capitalOf}(1), 1)) = 1 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & \text{if for every } x \in \mathbb{N}, \mathcal{B}(\text{locatedIn}(2x, x)) = 1,\ \text{and } \mathcal{B}(\text{locatedIn}(2, 1)) = 1 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & \text{if for every } x \in \mathbb{N}, (2x, x) \in \text{locatedIn}^{\mathcal{A}} \text{ and } (2, 1) \in \text{locatedIn}^{\mathcal{A}} \\ 0, & \text{otherwise} \end{cases}$$
$$= 1, \qquad \text{because } (2n, n) \in \text{locatedIn}^{\mathcal{B}} \text{ for every } n \in \mathbb{N}$$