

# CS 7810 - Knowledge Representation and Reasoning (for the Semantic Web)

## 08 - Tableau Algorithms for DLs

Adila Krisnadhi

Data Semantics Lab  
Wright State University, Dayton, OH

November 10, 2016

- 1 Basic Idea: Example from Propositional Logic
- 2 Satisfiability of  $\mathcal{ALC}$  Concepts
- 3 Satisfiability of  $\mathcal{ALC}$  Knowledge Bases

Materials in this presentation are adapted from:

- Sebastian Rudolph, “Tableau Procedures I”, slides for Foundations of Semantic Web Technologies course, TU Dresden, May 23, 2014.
- Sebastian Rudolph, “Tableau Procedures II”, slides for Foundations of Semantic Web Technologies course, TU Dresden, May 30, 2014.

- 1 Basic Idea: Example from Propositional Logic
- 2 Satisfiability of  $\mathcal{ALC}$  Concepts
- 3 Satisfiability of  $\mathcal{ALC}$  Knowledge Bases

- A concept is satisfiable if it has a model, i.e., there is an interpretation  $\mathcal{I}$  such that  $C^{\mathcal{I}} \neq \emptyset$
- Given a concept  $C$ , how do you decide if it is satisfiable?
  - So far: try to come up with an arbitrary model of  $C$ .
  - Can we automate it?
- Tableau algorithm: constructive decision procedure that tries to build models, if possible.
- Analogy from propositional logic:
  - Truth tables: enumerate exponentially many interpretations until finding a model
  - Tableau algorithm for propositional logic (can avoid checking exponentially many combinations)

Is the following formula satisfiable:  $(p \vee q) \rightarrow (\neg p \vee \neg q)$ ?

Negation in front of complex expressions difficult to handle, so reformulate:

$$(p \vee q) \rightarrow (\neg p \vee \neg q)$$

Is the following formula satisfiable:  $(p \vee q) \rightarrow (\neg p \vee \neg q)$ ?

Negation in front of complex expressions difficult to handle, so reformulate:

$$(p \vee q) \rightarrow (\neg p \vee \neg q)$$

$$\neg(p \vee q) \vee (\neg p \vee \neg q)$$

Is the following formula satisfiable:  $(p \vee q) \rightarrow (\neg p \vee \neg q)$ ?

Negation in front of complex expressions difficult to handle, so reformulate:

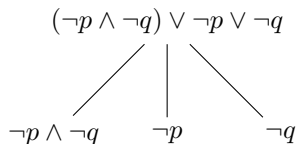
$$(p \vee q) \rightarrow (\neg p \vee \neg q)$$

$$\neg(p \vee q) \vee (\neg p \vee \neg q)$$

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$



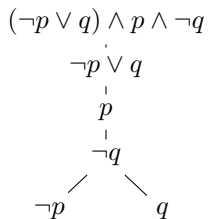
$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

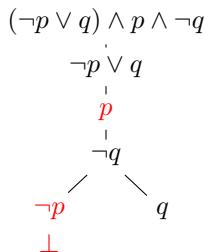


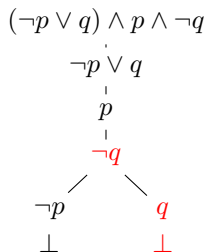


$$(\neg p \vee q) \wedge p \wedge \neg q$$

$$\begin{array}{c} (\neg p \vee q) \wedge p \wedge \neg q \\ \vdots \\ \neg p \vee q \\ \vdots \\ p \\ \vdots \\ \neg q \end{array}$$







- **complete branch**: (i) if  $p \wedge q$  in the branch, then so are  $p$  and  $q$ ; (ii) if  $p \vee q$  in the branch, then  $p$  or  $q$  or both are in the branch
- **closed branch**: contains an **atomic contradiction (clash)**
- **closed tableau**: all of its branches are closed
- **termination condition**: if every branch is either closed or complete
- tableau has an open and complete branch  $\rightsquigarrow$  formula is satisfiable
- from an open and complete branch, we can construct a model (see whiteboard)
- tableau is closed  $\rightsquigarrow$  formula is unsatisfiable



- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$(\neg p \vee q) \wedge p \wedge q$$

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{aligned} &(\neg p \vee q) \wedge p \wedge q \\ &\quad \neg p^{1a} \vee q^{1b} \\ &\quad p \\ &\quad q \end{aligned}$$

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{l} (\neg p \vee q) \wedge p \wedge q \\ \neg p^{1a} \vee q^{1b} \\ p \\ q \\ \neg p^{1a} \end{array}$$

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{l} (\neg p \vee q) \wedge p \wedge q \\ \neg p^{1a} \vee q^{1b} \\ p \\ q \\ \neg p^{1a} \\ \perp^{1a} \end{array}$$

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$(\neg p \vee q) \wedge p \wedge q$$

$$\neg p^{1a} \vee q^{1b}$$

$p$

$q$

~~$\neg p^{1a}$~~

~~$q^{1a}$~~

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c} (\neg p \vee q) \wedge p \wedge q \\ \neg p^{1a} \vee q^{1b} \\ p \\ q \\ \cancel{\neg p^{1a}} \\ \cancel{\neg p^{1a}} \\ q^{1b} \end{array}$$

↪ Found an open and complete branch.

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$(\neg p \vee q) \wedge p \wedge q$$

$$\neg p^{1a} \vee q^{1b}$$

$p$

$q$

~~$\neg p^{1a}$~~

~~$\neg p^{1a}$~~

$q^{1b}$

$$(\neg p \vee q) \wedge p \wedge \neg q$$

↪ Found an open and complete branch.

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{p^{1a}} \\
 q^{1b}
 \end{array}$$

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q
 \end{array}$$

↪ Found an open and complete branch.



- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{q^{1a}} \\
 q^{1b}
 \end{array}$$

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q \\
 \neg p^{1a}
 \end{array}$$

↪ Found an open and complete branch.

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 q^{1b}
 \end{array}$$

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q \\
 \neg p^{1a} \\
 \perp^{1a}
 \end{array}$$

↪ Found an open and complete branch.

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{q^{1a}} \\
 q^{1b}
 \end{array}$$

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q \\
 \cancel{\neg p^{1a}} \\
 \cancel{q^{1a}}
 \end{array}$$

↪ Found an open and complete branch.

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 q^{1b}
 \end{array}$$

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 q^{1b}
 \end{array}$$

⇒ Found an open and complete branch.

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 q^{1b}
 \end{array}$$

↪ Found an open and complete branch.

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 q^{1b} \\
 \perp^{1b}
 \end{array}$$

- mark disjunction with choice points, each corresponds to a branch
- all extensions of the branch due to such a choice are also marked
- when clash occurs, remove marked formulas and try next choice

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 q^{1b}
 \end{array}$$

↪ Found an open and complete branch.

$$\begin{array}{c}
 (\neg p \vee q) \wedge p \wedge \neg q \\
 \neg p^{1a} \vee q^{1b} \\
 p \\
 \neg q \\
 \cancel{\neg p^{1a}} \\
 \cancel{\perp^{1a}} \\
 \cancel{q^{1b}} \\
 \cancel{\perp^{1b}}
 \end{array}$$

↪ All branches are closed.

- 1 Basic Idea: Example from Propositional Logic
- 2 Satisfiability of  $\mathcal{ALC}$  Concepts
- 3 Satisfiability of  $\mathcal{ALC}$  Knowledge Bases

- Reasoning problem: “given a concept  $C$ , is  $C$  satisfiable?”
- We start with a simpler setting: knowledge base is empty  
 $\rightsquigarrow C$  is unsatisfiable if it is contradictory “by itself”
- tableau branch: finite set of atomic propositions of the form  $C(a)$ ,  $R(a, b)$   
(can be visualized as a graph involving elements of the universe)
- tableau: set of branches  $\rightsquigarrow$  set of “possible graphs”
- for each existential quantifier: introduce a new domain element
- for each universal quantifier: propagate filler concept expressions to neighboring elements.
- as in propositional tableau, negations must only appear in front of atomic concepts
- **clash** occurs if (i) both propositions of the form  $C(a)$  and  $\neg C(a)$  is in a branch; or (ii) proposition of the form  $\perp(a)$  is in a branch



$$\neg(C \sqcup D) \rightsquigarrow \neg C \sqcap \neg D$$

$$\neg(C \sqcap D) \rightsquigarrow \neg C \sqcup \neg D$$

$$\neg\neg C \rightsquigarrow C$$

$$\neg(\forall R.C) \rightsquigarrow \exists R.\neg C$$

$$\neg(\exists R.C) \rightsquigarrow \forall R.\neg C$$

$$\neg(\leq n R.C) \rightsquigarrow \geq (n + 1) R.C$$

$$\neg(\geq n R.C) \rightsquigarrow \leq (n - 1) R.C, \quad n \geq 1$$

$$\neg(\geq 0 R.C) \rightsquigarrow \perp$$

$$(\geq 0 R.C) \rightsquigarrow \top$$

- apply the above rules exhaustively (until none can be applied)
- result: equivalent concept in **negation normal form (NNF)**
- example:  $\neg(\exists R.\neg C \sqcap \forall S.(\neg D \sqcup E)) \equiv \forall R.C \sqcup \exists S.(D \sqcap \neg E)$

**Data structure:** labeled graph where  $\mathbf{V}$  is the set of nodes,  $\mathbf{E}$  is the set of edges (pairs of nodes),  $\mathbf{L}(v)$  is the set of labels of a node  $v$ , and  $\mathbf{L}(v, v')$  is the set of labels of the edge from node  $v$  to node  $v'$ .

**Input:**  $\mathcal{ALC}$  concept  $C$  in NNF.

**Initialization:**  $\mathbf{V} := \{v_0\}$ ,  $\mathbf{E} := \emptyset$ , and  $\mathbf{L}(v_0) := \{C\}$

Extend the graph by applying any applicable tableau rules until no more rules can be applied.

- $\sqcap$ -rule: if there is a node  $v$  with  $D \sqcap E \in \mathbf{L}(v)$  and  $\{D, E\} \not\subseteq \mathbf{L}(v)$ , then set  $\mathbf{L}(v) := \mathbf{L}(v) \cup \{D, E\}$
- $\sqcup$ -rule: if there is a node  $v$  with  $D \sqcup E \in \mathbf{L}(v)$  and  $\{D, E\} \cap \mathbf{L}(v) = \emptyset$ , then choose one of  $X \in \{D, E\}$  nondeterministically and set  $\mathbf{L}(v) := \mathbf{L}(v) \cup \{X\}$
- $\exists$ -rule: if there is a node  $v$  with  $\exists R.D \in \mathbf{L}(v)$  and there is no node  $v'$  such that  $\langle v, v' \rangle \in E$  and  $D \in \mathbf{L}(v')$ , then create a new node  $v'$ , set  $\mathbf{V} := \mathbf{V} \cup \{v'\}$ ,  $\mathbf{E} := \mathbf{E} \cup \{\langle v, v' \rangle\}$ ,  $\mathbf{L}(v') := \{D\}$ , and  $\mathbf{L}(v, v') := \{R\}$
- $\forall$ -rule: if there are nodes  $v, v'$  with  $\langle v, v' \rangle \in E$ ,  $R \in \mathbf{L}(v, v')$ ,  $\forall R.D \in \mathbf{L}(v)$ , and  $D \notin \mathbf{L}(v')$ , then set  $\mathbf{L}(v') := \mathbf{L}(v') \cup \{D\}$

**Output:** “satisfiable” if we can construct a clash-free tableau where no more rules can be applied. Otherwise, “unsatisfiable”

Note: rule applications exhibit “don’t care” nondeterminism; choice of disjunction exhibits “don’t know” nondeterminism

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

$v_0$

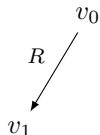
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

$v_0$

$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)), \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

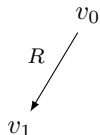
Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B\}$$

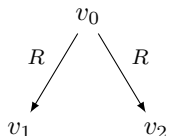
Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

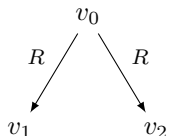


$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B \}$$

$$\mathbf{L}(v_2) = \{ \neg A \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



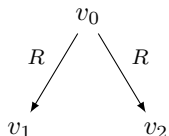
$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B \}$$

$$\mathbf{L}(v_2) = \{ \neg A \}$$



Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

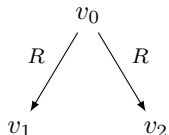


$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

$$\mathbf{L}(v_2) = \{ \neg A \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

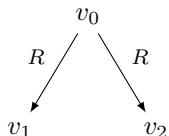


$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

$$\mathbf{L}(v_2) = \{ \neg A, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

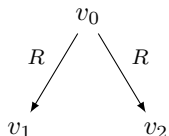


$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

$$\mathbf{L}(v_2) = \{ \neg A, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

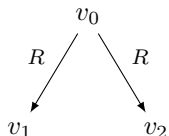


$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

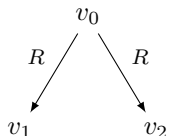


$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

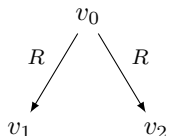


$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), A\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

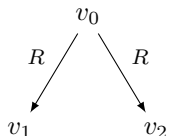


$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), A\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



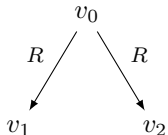
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{\exists R.B}\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$



Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

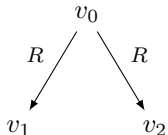


$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{\exists R.B} \}$$

$$\mathbf{L}(v_2) = \{ \neg A, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$

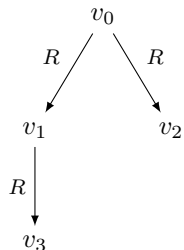


$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{\exists R.B}\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



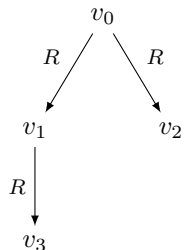
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{A}, \\ \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



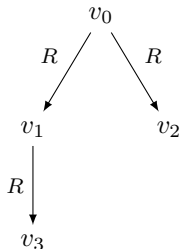
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



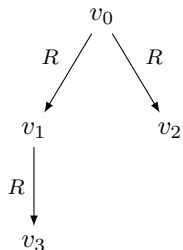
$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)), \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \exists R.B \}$$

$$\mathbf{L}(v_2) = \{ \neg A, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

$$\mathbf{L}(v_3) = \{ B, \neg B \sqcup A \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



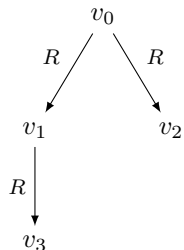
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{\exists R.B}\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B, \neg B \sqcup A\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



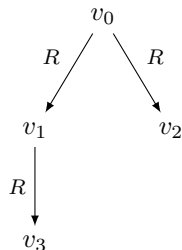
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)), \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B, \neg B \sqcup A, \neg B\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

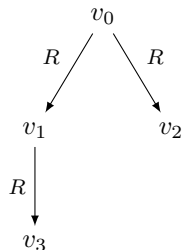
$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{A}, \\ \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B, \neg B \sqcup A, \neg B\}$$



Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



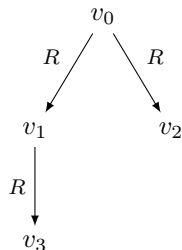
$$\mathbf{L}(v_0) = \{ \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)), \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \}$$

$$\mathbf{L}(v_1) = \{ A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{A}, \\ \exists R.B \}$$

$$\mathbf{L}(v_2) = \{ \neg A, \neg A \sqcap \forall R.(\neg B \sqcup A) \}$$

$$\mathbf{L}(v_3) = \{ B, \neg B \sqcup A, \cancel{B} \}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



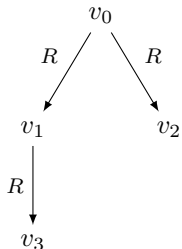
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)) \\ \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{A}, \\ \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B, \neg B \sqcup A, \cancel{B}, A\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



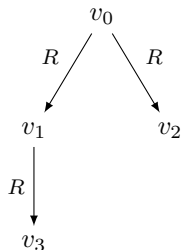
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)), \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B, \neg B \sqcup A, A\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



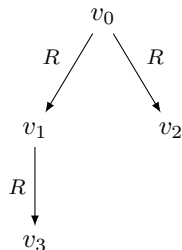
$$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)), \exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$$

$$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \cancel{A}, \exists R.B\}$$

$$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A), \forall R.(\neg B \sqcup A)\}$$

$$\mathbf{L}(v_3) = \{B, \neg B \sqcup A, \cancel{B}, A\}$$

Input:  $\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$



$\mathbf{L}(v_0) = \{\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$   
 $\exists R.(A \sqcup \exists R.B), \exists R.\neg A, \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))\}$

$\mathbf{L}(v_1) = \{A \sqcup \exists R.B, \neg A \sqcap \forall R.(\neg B \sqcup A), \neg A, \forall R.(\neg B \sqcup A), \exists R.B\}$

$\mathbf{L}(v_2) = \{\neg A, \neg A \sqcap \forall R.(\neg B \sqcup A), \forall R.(\neg B \sqcup A)\}$

$\mathbf{L}(v_3) = \{B, \neg B \sqcup A, A\}$

Since the complete tableau is clash-free, the output is “satisfiable”

$\rightsquigarrow$  the input concept is satisfiable, and we can construct a model (next slide)

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$



A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\} \quad (\forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$



A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\} \quad (\forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$(\neg A \sqcap \forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\} \quad (\forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$(\neg A \sqcap \forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)))^{\mathcal{I}} = \{v_0, v_2, v_3\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\} \quad (\forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$(\neg A \sqcap \forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)))^{\mathcal{I}} = \{v_0, v_2, v_3\}$$

$$(A \sqcup \exists R.B)^{\mathcal{I}} = \{v_1, v_3\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\} \quad (\forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$(\neg A \sqcap \forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)))^{\mathcal{I}} = \{v_0, v_2, v_3\}$$

$$(A \sqcup \exists R.B)^{\mathcal{I}} = \{v_1, v_3\} \quad (\exists R.(A \sqcup \exists R.B))^{\mathcal{I}} = \{v_0, v_1\}$$

A model  $\mathcal{I}$  for  $C := \exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A))$  is as follows:

$$\Delta^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$A^{\mathcal{I}} = \{v_3\}$$

$$B^{\mathcal{I}} = \{v_3\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_3 \rangle\}$$

The following are easy to verify by the semantics:

$$(\neg A)^{\mathcal{I}} = (\neg B)^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\exists R.B)^{\mathcal{I}} = \{v_1\} \quad (\exists R.\neg A)^{\mathcal{I}} = \{v_0\}$$

$$(\neg B \sqcup A)^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\} \quad (\forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2, v_3\}$$

$$(\neg A \sqcap \forall R.(\neg B \sqcup A))^{\mathcal{I}} = \{v_0, v_1, v_2\} \quad (\forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)))^{\mathcal{I}} = \{v_0, v_2, v_3\}$$

$$(A \sqcup \exists R.B)^{\mathcal{I}} = \{v_1, v_3\} \quad (\exists R.(A \sqcup \exists R.B))^{\mathcal{I}} = \{v_0, v_1\}$$

$$(\exists R.(A \sqcup \exists R.B) \sqcap \exists R.\neg A \sqcap \forall R.(\neg A \sqcap \forall R.(\neg B \sqcup A)))^{\mathcal{I}} = \{v_0\}$$

Since  $C^{\mathcal{I}} \neq \emptyset$ ,  $C$  is thus satisfiable.

- termination:
  - the number of nested quantifiers decrease in every node generated
  - every node is labeled only with subformulas of the input concept
  - the input concept has only polynomially many subformulas
- soundness:
  - if the output is “satisfiable”, then we can construct a model of the input concept, which implies that the input concept is indeed satisfiable
- completeness:
  - if the input concept is satisfiable, then it has a model, and this model can be used to construct a clash-free tableau for the concept.

## Theorem

- 1 The tableau algorithm for  $\mathcal{ALC}$  concepts terminates for every input
- 2 If the output is “satisfiable”, then the input concept is satisfiable
- 3 If the input concept is satisfiable, then the output is “satisfiable”

## Corollary

Every  $\mathcal{ALC}$  concept  $C$  has the following properties:

- 1 **finite model property**: if  $C$  has a model, then it also has a finite model (i.e., has only finitely many universe elements)
  - 2 **tree model property**: if  $C$  has a model, then it also has a tree-shaped model
- the finite and tree-shaped model above can be obtained by the model construction from a clash-free tableau
  - finiteness and/or tree-shapedness may no longer hold in the presence of knowledge bases (i.e., not just concepts)

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

$v_0$

$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \exists R.A \sqcup \exists R.\neg B, \forall R.(\neg A \sqcap B)\}$$



Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

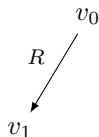
Note: Formulas due to picking a choice point are marked with underscore.

$v_0$

$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \underline{\exists R.A} \sqcup \exists R.\neg B, \forall R.(\neg A \sqcap B), \underline{\exists R.A}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

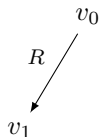


$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \underline{\exists R.A} \sqcup \exists R.\neg B, \forall R.(\neg A \sqcap B), \underline{\exists R.A}\}$$

$$\mathbf{L}(v_1) = \{\underline{A}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

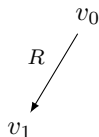


$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \underline{\exists R.A} \sqcup \exists R.\neg B, \forall R.(\neg A \sqcap B), \underline{\exists R.A}\}$$

$$\mathbf{L}(v_1) = \{\underline{A}, \underline{\neg A \sqcap B}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

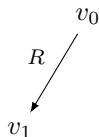


$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \underline{\exists R.A} \sqcup \exists R.\neg B, \forall R.(\neg A \sqcap B), \underline{\exists R.A}\}$$

$$\mathbf{L}(v_1) = \{\underline{A}, \underline{\neg A} \sqcap B, \underline{\neg A}, \underline{B}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.



$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \underline{\exists R.A} \sqcup \exists R.\neg B, \forall R.(\neg A \sqcap B), \underline{\exists R.A}\}$$

$$\mathbf{L}(v_1) = \{\underline{A}, \underline{\neg A \sqcap B}, \cancel{A}, B\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

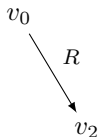
Note: Formulas due to picking a choice point are marked with underscore.

$v_0$

$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \underline{\exists R.A \sqcup \exists R.\neg B}, \forall R.(\neg A \sqcap B), \underline{\exists R.\neg B}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

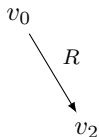


$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \exists R.A \sqcup \underline{\exists R.\neg B}, \forall R.(\neg A \sqcap B), \underline{\exists R.\neg B}\}$$

$$\mathbf{L}(v_2) = \{\underline{\neg B}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.



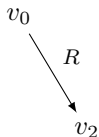
$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \exists R.A \sqcup \underline{\exists R.\neg B}, \underline{\forall R.(\neg A \sqcap B)}, \underline{\exists R.\neg B}\}$$

$$\mathbf{L}(v_2) = \{\underline{\neg B}, \underline{\neg A \sqcap B}\}$$



Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

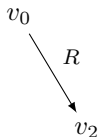


$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \exists R.A \sqcup \underline{\exists R.\neg B}, \forall R.(\neg A \sqcap B), \underline{\exists R.\neg B}\}$$

$$\mathbf{L}(v_2) = \{\underline{\neg B}, \underline{\neg A \sqcap B}, \underline{\neg A}, \underline{B}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.

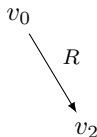


$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \exists R.A \sqcup \underline{\exists R.\neg B}, \forall R.(\neg A \sqcap B), \underline{\exists R.\neg B}\}$$

$$\mathbf{L}(v_2) = \{\underline{\neg B}, \underline{\neg A \sqcap B}, \underline{\neg A}, \cancel{B}\}$$

Input:  $(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B)$

Note: Formulas due to picking a choice point are marked with underscore.



$$\mathbf{L}(v_0) = \{(\exists R.A \sqcup \exists R.\neg B) \sqcap \forall R.(\neg A \sqcap B), \exists R.A \sqcup \underline{\exists R.\neg B}, \forall R.(\neg A \sqcap B), \underline{\exists R.\neg B}\}$$

$$\mathbf{L}(v_2) = \{\underline{\neg B}, \underline{\neg A \sqcap B}, \underline{\neg A}, \cancel{B}\}$$

All choice points lead to a clash  $\rightsquigarrow$  the concept is unsatisfiable.

- 1 Basic Idea: Example from Propositional Logic
- 2 Satisfiability of  $\mathcal{ALC}$  Concepts
- 3 Satisfiability of  $\mathcal{ALC}$  Knowledge Bases

Instead of concept satisfiability, we consider knowledge base satisfiability.

## Knowledge Base Satisfiability

Given a knowledge base  $\mathcal{K}$ , is  $\mathcal{K}$  satisfiable?

Note that a knowledge base is the union of a TBox, an ABox, and an RBox. For  $\mathcal{ALC}$ , RBox is always empty.

If we have a decision procedure (i.e., algorithm) for KB satisfiability, then we could use it to solve other DL basic reasoning problems.

Below,  $\mathcal{K}$  is a knowledge base,

$c, c_0, \dots, c_n$  are fresh individual names not occurring in  $\mathcal{K}$ ,

$U$  is the universal role (usable if the logic allows it –  $\mathcal{ALC}$  does not!),

$a, b$  are individual names (may or may not occur in  $\mathcal{K}$ ),

$C, D$  are concepts,  $R, R_1, \dots, R_n$  are roles/properties.

### 1 Axiom entailment:

- $\mathcal{K} \models C \sqsubseteq D$  iff  $\mathcal{K} \cup \{(C \sqcap \neg D)(c)\}$  is unsatisfiable
- $\mathcal{K} \models C \sqsubseteq D$  iff  $\mathcal{K} \cup \{\top \sqsubseteq \exists U.(C \sqcap \neg D)\}$  is unsatisfiable
- $\mathcal{K} \models C(a)$  iff  $\mathcal{K} \cup \{\neg C(a)\}$  is unsatisfiable
- $\mathcal{K} \models R(a, b)$  iff  $\mathcal{K} \cup \{\neg R(a, b)\}$  is unsatisfiable
- $\mathcal{K} \models \neg R(a, b)$  iff  $\mathcal{K} \cup \{R(a, b)\}$  is unsatisfiable
- $\mathcal{K} \models \text{Dis}(R_1, R_2)$  iff  $\mathcal{K} \cup \{R_1(c_1, c_2), R_2(c_1, c_2)\}$  is unsatisfiable
- $\mathcal{K} \models R_1 \circ \dots \circ R_n \sqsubseteq R$  iff  $\mathcal{K} \cup \{\neg R(c_0, c_n), R_1(c_0, c_1), \dots, R_n(c_{n-1}, c_n)\}$  is unsatisfiable

2 **Concept (un)satisfiability:**

$C$  is unsatisfiable w.r.t.  $\mathcal{K}$  iff  $\mathcal{K} \models C \sqsubseteq \perp$  iff  $\mathcal{K} \cup \{C(c)\}$  is unsatisfiable.  
 $\rightsquigarrow$  Thus,  $C$  is satisfiable w.r.t  $\mathcal{K}$  iff  $\mathcal{K} \cup \{C(c)\}$  is satisfiable.

3 **Concept subsumption:**

$C$  is subsumed by  $D$  w.r.t.  $\mathcal{K}$  iff  $\mathcal{K} \models C \sqsubseteq D$  iff  $\mathcal{K} \cup \{(C \sqcap \neg D)(c)\}$  is unsatisfiable iff  $\mathcal{K} \cup \{\top \sqsubseteq \exists U.(C \sqcap \neg D)\}$  is unsatisfiable

4 **Instance checking:**

An individual  $a$  is an instance of a concept  $C$  w.r.t  $\mathcal{K}$  iff  $\mathcal{K} \models C(a)$

Tableau algorithm for deciding knowledge base satisfiability is obtained by modifying/extending the tableau algorithm for deciding concept satisfiability as follows:

- Accommodating ABox  $\rightsquigarrow$  modify the initialization phase by using information from the ABox
- Accommodating TBox  $\rightsquigarrow$  internalize/compress the TBox and add a tableau rule special for TBox

Other tableau rules ( $\sqcap$ ,  $\sqcup$ ,  $\exists$ ,  $\forall$ ) as well as the definition of clash stay the same.



We accommodate the ABox by modifying the initialization:

For ABox  $\mathcal{A}$  part of the input, initialize the tableau graph  $G = \langle \mathbf{V}, \mathbf{E}, \mathbf{L} \rangle$ :

- Initialize the set of nodes  $\mathbf{V}$  to contain a node  $v_a$  for every individual name  $a$  occurring in  $\mathcal{A}$
- Initialize node labels  $\mathbf{L}(v_a) := \{C \mid C(a) \in \mathcal{A}\}$
- For every role assertion  $R(a, b)$ , initialize the set of edges  $\mathbf{E}$  to contain an edge  $\langle v_a, v_b \rangle$  and the edge label  $\mathbf{L}(v_a, v_b)$  to contain  $R$ .

If  $\mathcal{A}$  is empty, we set  $\mathbf{V} := \{v_0\}$  for a fresh node  $v_0$  and  $\mathbf{E} := \emptyset$  and  $\mathbf{L}(v_0) := \emptyset$ .

Afterwards, the tableau rules can be applied to the graph initialized as above.

- Concept equivalence  $C \equiv D$  are replaced with  $C \sqsubseteq D$  and  $D \sqsubseteq C$
- Every GCI  $C \sqsubseteq D$  is equivalent to  $\top \sqsubseteq \neg C \sqcup D$

The TBox containing  $n$  GCIs:

$$\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$$

can be compressed/**internalized** into the following equivalent TBox containing only a single axiom:

$$\mathcal{T}' = \{\top \sqsubseteq \prod_{1 \leq i \leq n} (\neg C_i \sqcup D_i)\}$$

Denote the NNF of the right-hand side of the GCI in  $\mathcal{T}'$  as the concept  $C_{\mathcal{T}}$ .

- Assuming the TBox is internalized, we could use the  $\mathcal{T}$ -rule:

$\mathcal{T}$ -rule: For an arbitrary node  $v$  such that  $C_{\mathcal{T}} \notin \mathbf{L}(v)$ , set  
 $\mathbf{L}(v) := \mathbf{L}(v) \cup \{C_{\mathcal{T}}\}$

- Assuming the TBox is internalized, we could use the  $\mathcal{T}$ -rule:

$\mathcal{T}$ -rule: For an arbitrary node  $v$  such that  $C_{\mathcal{T}} \notin \mathbf{L}(v)$ , set  
 $\mathbf{L}(v) := \mathbf{L}(v) \cup \{C_{\mathcal{T}}\}$

But there is a potential problem ...

Consider TBox  $\mathcal{T} = \{\top \sqsubseteq A, A \sqsubseteq \exists R.A\}$ . Is  $A$  satisfiable given  $\mathcal{T}$ ? (That is, is there a model of both  $A$  and  $\mathcal{T}$ ?)

- Assuming the TBox is internalized, we could use the  $\mathcal{T}$ -rule:

$\mathcal{T}$ -rule: For an arbitrary node  $v$  such that  $C_{\mathcal{T}} \notin \mathbf{L}(v)$ , set  
 $\mathbf{L}(v) := \mathbf{L}(v) \cup \{C_{\mathcal{T}}\}$

But there is a potential problem ...

Consider TBox  $\mathcal{T} = \{\top \sqsubseteq A, A \sqsubseteq \exists R.A\}$ . Is  $A$  satisfiable given  $\mathcal{T}$ ? (That is, is there a model of both  $A$  and  $\mathcal{T}$ ?)

Termination is not guaranteed!

Reason:

- Assuming the TBox is internalized, we could use the  $\mathcal{T}$ -rule:

$\mathcal{T}$ -rule: For an arbitrary node  $v$  such that  $C_{\mathcal{T}} \notin \mathbf{L}(v)$ , set  
 $\mathbf{L}(v) := \mathbf{L}(v) \cup \{C_{\mathcal{T}}\}$

But there is a potential problem ...

Consider TBox  $\mathcal{T} = \{\top \sqsubseteq A, A \sqsubseteq \exists R.A\}$ . Is  $A$  satisfiable given  $\mathcal{T}$ ? (That is, is there a model of both  $A$  and  $\mathcal{T}$ ?)

Termination is not guaranteed!

Reason: the quantifier depth does not necessarily decrease for newly introduced child nodes.

What do we do?

- Assuming the TBox is internalized, we could use the  $\mathcal{T}$ -rule:

$\mathcal{T}$ -rule: For an arbitrary node  $v$  such that  $C_{\mathcal{T}} \notin \mathbf{L}(v)$ , set  
 $\mathbf{L}(v) := \mathbf{L}(v) \cup \{C_{\mathcal{T}}\}$

But there is a potential problem ...

Consider TBox  $\mathcal{T} = \{\top \sqsubseteq A, A \sqsubseteq \exists R.A\}$ . Is  $A$  satisfiable given  $\mathcal{T}$ ? (That is, is there a model of both  $A$  and  $\mathcal{T}$ ?)

Termination is not guaranteed!

Reason: the quantifier depth does not necessarily decrease for newly introduced child nodes.

What do we do?  $\rightsquigarrow$  we should recognize “cycles” (repeated node labelings)

Let  $G = \langle \mathbf{V}, \mathbf{E}, \mathbf{L} \rangle$  be the tableau graph/tree.

A node  $v \in \mathbf{V}$  **directly blocks** a node  $v' \in \mathbf{V}$ , if:

- 1  $v'$  is reachable from  $v$ ,
- 2  $\mathbf{L}(v') \subseteq \mathbf{L}(v)$ , and
- 3 there is no directly blocking node  $v''$  such that  $v'$  is reachable from  $v''$

A node  $v'$  is **blocked** if either  $v'$  is directly blocked node or there is a directly blocked node  $w$  such that  $v'$  is reachable from  $w$ .

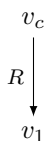
The  $\exists$ -rule can only be applied to nodes that are **NOT blocked**.



Is  $A$  satisfiable with respect to the TBox  $\mathcal{T} = \{A \sqsubseteq \exists R.A\}$ ?

**Answer:** First,  $C_{\mathcal{T}} = \neg A \sqcup \exists R.A$ . Also,  $A$  is satisfiable w.r.t  $\mathcal{T}$  iff  $\mathcal{T} \cup \{A(c)\}$  is satisfiable.

The clash-free tableau is:



$$\mathbf{L}(v_c) = \{A, \neg A \sqcup \exists R.A, \exists R.A\}$$

$$\mathbf{L}(v_1) = \{A, \neg A \sqcup \exists R.A, \exists R.A\}$$

note:  $v_1$  is directly blocked by  $v_c$

- Blocked nodes do not represent elements in the model.
- For each edge from  $v$  to  $v'$ , if  $v'$  is directly blocked (by some node, say  $w$ ), then the model would have an “edge” from  $v$  to  $w$  instead.
- This model is finite  $\rightsquigarrow$  finite model property holds.
- But the model may not be tree-shaped.

The tableau from the previous slide gives us the following model of  $A$  and  $\mathcal{T}$ .

$$\Delta^{\mathcal{I}} = \{v_0\}$$

$$A^{\mathcal{I}} = \{v_0\}$$

$$R^{\mathcal{I}} = \{\langle v_0, v_0 \rangle\}$$

- Is  $A$  satisfiable with respect to  $\mathcal{T} = \{A \sqsubseteq \exists R.A \sqcap \exists S.B\}$ ?
- Is  $A$  satisfiable with respect to  $\mathcal{T} = \{A \sqsubseteq \exists R.B, B \sqsubseteq D \sqcap \forall S.B, D \sqsubseteq \exists S.C, B \sqcap C \sqsubseteq \perp\}$ ?
- Is  $A$  satisfiable with respect to  $\mathcal{T} = \{A \sqsubseteq B \sqcap \exists R.C, B \equiv C \sqcup D, C \sqsubseteq \exists R.D, \exists R.B \sqsubseteq A\}$ ?

For each of the above example, if the answer is yes, give a model of  $\mathcal{T}$  that satisfies  $A$ .